Numerical technique for studying stochastic resonance

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A numerical technique is proposed to study the stochastic resonance (SR) phenomenon. The proposed technique allows one to find characteristics of SR faster than by the usual approach. The signal-to-noise ratio and phase shifts for a system of noisy coupled oscillators are simulated. [S1063-651X(98)00704-1]

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I. INTRODUCTION

The term ''stochastic resonance'' (SR) is historically used in connection with a variety of effects attributable to the interaction between a periodic applied force and noise in nonlinear systems. As a survey on SR, one can use the proceedings of workshops [1,2] or the review [3].

In recent papers [4–7] the authors have investigated SR in large arrays (up to 512 elements) of noisy coupled oscillators. The phenomenon was named in [4] array enhanced stochastic resonance (AESR). In addition to the common features of SR, AESR demonstrates a spatiotemporal synchronization and there is an additional design parameter, the coupling strength, which essentially affects the behavior of SR characteristics. It was experimentally shown in [7] that the signal-to-noise ratio of the output signal of a single diode resonator can be significantly improved by coupling it diffusively into an array of resonators. AESR was studied analytically in some limit cases (see, e.g., [8,5]), but the basic tool for its investigation is numerical simulation of a system of stochastic differential equations (SDEs).

To calculate the characteristics describing SR, one must integrate the system on long time intervals and simulate a sufficiently large number of independent realizations. Main characteristics of SR (e.g., the signal-to-noise ratio) are expectations of functionals of the SDE solution. It is known [9–11] that weak numerical methods are sufficient to calculate such quantities and are quite simple for realization and efficient. Special powerful weak methods for SDEs with relatively small noise are proposed in [12]. Applying these methods, we propose a numerical technique here that allows us to study properties of SR faster than by the usual approach. We demonstrate the technique on a simple model, but it is also valid for more complicated systems.

In Sec. II the numerical technique for calculating the signal-to-noise ratio and phase shifts is proposed. Section III contains numerical results for the array of noisy coupled oscillators. We confirm by the numerical experiment that the signal-to-noise ratio (SNR) has extremal behavior with an increase of both noise intensity and coupling and that the effect is improved with the growth of the array length. These results are in full agreement with the previous papers on AESR [4,5]. We also show that there is a critical length of array such that a further increase of the array length does not lead to an improved SNR. The experiments on phase shifts display their extremal behavior with the growth of noise intensity and coupling.

II. DESCRIPTION OF THE NUMERICAL TECHNIQUE

Following [4,5], we consider the system of stochastic differential equations

$$dX^{(i)}=[aX^{(i)}-bX^{(i)^3}+A\ \sin(\Omega t+\varphi)+c(X^{(i+1)}-2X^{(i)}+X^{(i-1)})]dt+\varepsilon dW_i(t),$$
$$i=1,2,\ldots,n, X^{(0)}\equiv X^{(1)}, X^{(n+1)}\equiv X^{(n)}, X(0)=X_0, t\in[0,T], \quad (1)$$

where $X=(X^{(1)},X^{(2)},\ldots,X^{(n)})$ is an n-dimensional vector and $W_i(t), i=1,\ldots,n$, are independent standard Wiener processes. The system (1) describes a one-dimensional array (chain) of overdamped driven nonlinear oscillators coupled linearly to their nearest neighbors. To ensure a bistable potential (two-state points), the coefficients a and b must be positive. The phase φ is taken as a uniformly distributed random variable on the interval $[0,2\pi]$ and the coupling parameter $c\geqslant 0$.

Let us recall some definitions from the theory of random processes (see, e.g., [13,14]), which we use below. A random process $\xi(t)$, $\xi(t)\in R^n$, is a stationary one if it has two first moments $E\xi(t)$ and $E|\xi(t)|^2$ and

$$E\xi(t)=E\xi(t+\tau)=\text{const},$$
$$\text{cov}(\xi(t),\xi(s))=\text{cov}(\xi(t+\tau),\xi(s+\tau)) \quad \text{for any } t,s,\tau, \quad (2)$$

where $\text{cov}(\xi(t),\xi(s))$ is the covariation matrix

$$\text{cov}(\xi(t),\xi(s))=E\xi(t)\xi(s)^T-E\xi(t)E\xi(s)^T.$$

A random process $\xi(t)$ is a periodic one with a period T_0 if

$$E\xi(t)=E\xi(t+T_0),$$
$$\text{cov}(\xi(t),\xi(s))=\text{cov}(\xi(t+T_0),\xi(s+T_0)) \quad \text{for any } t,s. \quad (3)$$

In accordance with Theorem 5.2 in [13], Chap. 3, there is a solution $X(t)$ of the system (1) at each fixed φ, which is a periodic Markov process with the period $2\pi/\Omega$. The peri-

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odic process can be converted into a stationary one by the following shift of time [15,13]: If θ is a random variable, distributed uniformly on $[0,2\pi/\Omega]$ and independent on $X(t)$, then the process $\eta(t)=X(t+\theta)$ is stationary. Consequently, there is a stationary solution $X(t)$ of the system (1) in the case of uniformly distributed on $[0,2\pi]$, independent on $X(t)$ random phase φ (see the physical background of such a distribution of the phase φ, e.g., in [16,3]). Due to Theorem 7.1 in [13], Chap. 4, there is a unique stationary Markov process $X(t)$ corresponding to the system (1), and under any initial distribution of X_0 the solutions of Eq. (1) converge to this stationary process in the weak sense as $t\to\infty$.

Let us consider a constituent oscillator of the chain (e.g., the middle one) described by Eq. (1). Below we shall denote a constituent component (oscillator) $X^{(l)}, l\in\{1,2,\ldots,n\}$, of the vector $X(t)\equiv(X^{(1)},\ldots,X^{(n)})$ by $X_c(t)$ and its correlation function by $K(\tau)$. In our case the correlation function $K(\tau)$ has the form (cf. [15])

$$K(\tau)=K_0(\tau)+2\sum_{k=1}^{\infty}\alpha_{2k-1}\cos(2k-1)\Omega\tau, \tag{4}$$

where $K_0(\tau)$ goes quickly to zero.

According to the definition, the spectral function $F(\omega)$ can be written as

$$F(\omega)=\frac{1}{\pi}\int_0^{\infty}K(\tau)\frac{\sin\omega\tau}{\tau}d\tau, \quad \omega\in[0,\infty). \tag{5}$$

Because of Eq. (4), we get

$$F(\omega)=F_0(\omega)+\sum_{k=1}^{\infty}\alpha_{2k-1}\chi[\omega-(2k-1)\Omega],$$

where

$$F_0(\omega)=\frac{1}{\pi}\int_0^{\infty}K_0(\tau)\frac{\sin\omega\tau}{\tau}d\tau, \quad \chi(x)=\begin{cases}0, & x<0\\ 1/2, & x=0\\ 1, & x>0.\end{cases}$$

Consequently, the spectral density $S(\omega)$ is equal to

$$S(\omega)=S_0(\omega)+\sum_{k=1}^{\infty}\alpha_{2k-1}\delta(\omega-(2k-1)\Omega) \tag{6}$$

and

$$S_0(\omega)=\frac{dF_0(\omega)}{d\omega}=\frac{1}{\pi}\int_0^{\infty}K_0(\tau)\cos\omega\tau d\tau.$$

Thus, as it is generally known, the spectrum $S(\omega)$ for a system like Eq. (1) consists of a broadband noise background and δ-function spikes at frequencies $\omega=(2k-1)\Omega$, $k=1,2,\ldots$.

A. Numerical method

Let us introduce an equidistant discretization Δ_N of the time interval $[0,T]$: $\Delta_N=\{t_i: i=0,1,\ldots,N;\ 0=t_0<t_1<\cdots<t_N=T\}$; the time step $h=t_{i+1}-t_i$. To simulate the system (1), which is a system with additive noises, we take the fully Runge-Kutta weak method with the error $O(h^4+\varepsilon^4h^2)$ from [12]. In the case of the system (1) this method takes the form

$$\begin{aligned}X_{k+1}^{(i)}=&X_k^{(i)}+\varepsilon h^{1/2}\xi_k^{(i)}+(k_1^{(i)}+2k_2^{(i)}+2k_3^{(i)}+k_4^{(i)})/6\\&+\varepsilon h^{3/2}a\,\eta_k^{(i)}-hb[(X_k^{(i)}+\varepsilon h^{1/2}\eta_k^{(i)})^3-(X_k^{(i)}\\&-\varepsilon h^{1/2}\eta_k^{(i)})^3]/2+\varepsilon h^{3/2}c(\eta_k^{(i+1)}-2\eta_k^{(i)}+\eta_k^{(i-1)}),\\&i=1,\ldots,n,\quad k=0,1,\ldots,N,\end{aligned} \tag{7}$$

where $X_k=(X_k^{(1)},X_k^{(2)},\ldots,X_k^{(n)})$ is the approximation of the solution $X(t_k)$ to system (1) and

$$\begin{aligned}k_1^{(i)}=&h[aX_k^{(i)}-bX_k^{(i)^3}+Av_k+c(X_k^{(i+1)}-2X_k^{(i)}+X_k^{(i-1)})],\\k_2^{(i)}=&h\{a(X_k^{(i)}+k_1^{(i)}/2)-b(X_k^{(i)}+k_1^{(i)}/2)^3+A(v_k+l_1/2)\\&+c[(X_k^{(i+1)}+k_1^{(i+1)}/2)-2(X_k^{(i)}+k_1^{(i)}/2)\\&+(X_k^{(i-1)}+k_1^{(i-1)}/2)]\},\\k_3^{(i)}=&h\{a(X_k^{(i)}+\varepsilon h^{1/2}\xi_k^{(i)}+k_2^{(i)}/2)-b(X_k^{(i)}+\varepsilon h^{1/2}\xi_k^{(i)}\\&+k_2^{(i)}/2)^3+A(v_k+l_2/2)+c[(X_k^{(i+1)}+\varepsilon h^{1/2}\xi_k^{(i+1)}\\&+k_2^{(i+1)}/2)-2(X_k^{(i)}+\varepsilon h^{1/2}\xi_k^{(i)}+k_2^{(i)}/2)+(X_k^{(i-1)}\\&+\varepsilon h^{1/2}\xi_k^{(i-1)}+k_2^{(i-1)}/2)]\},\\k_4^{(i)}=&h\{a(X_k^{(i)}+\varepsilon h^{1/2}\xi_k^{(i)}+k_3^{(i)})-b(X_k^{(i)}+\varepsilon h^{1/2}\xi_k^{(i)}+k_3^{(i)})^3\\&+A(v_k+l_3)+c[(X_k^{(i+1)}+\varepsilon h^{1/2}\xi_k^{(i+1)}+k_3^{(i+1)})\\&-2(X_k^{(i)}+\varepsilon h^{1/2}\xi_k^{(i)}+k_3^{(i)})+(X_k^{(i-1)}+\varepsilon h^{1/2}\xi_k^{(i-1)}\\&+k_3^{(i-1)})]\},\\&k_j^{(0)}\equiv k_j^{(1)},\quad k_j^{(n+1)}\equiv k_j^{(n)},\quad j=1,\ldots,4,\\&X_k^{(0)}\equiv X_k^{(1)},\quad X_k^{(n+1)}\equiv X_k^{(n)},\quad X_0=X(0).\end{aligned} \tag{8}$$

To simulate $\sin(\Omega t+\varphi)$ of Eq. (1), we introduce the system

$$du=-\Omega v\,dt,$$

$$dv=\Omega u dt,$$

$$u(0)=u_0=\cos\varphi,\quad v(0)=v_0=\sin\varphi.$$

This system is approximated by

$$u_{k+1}=u_k+(m_1+2m_2+2m_3+m_4)/6,$$

$$v_{k+1}=v_k+(l_1+2l_2+2l_3+l_4)/6,$$

where

$$m_1=-h\Omega v_k,\quad l_1=h\Omega u_k,\quad m_2=-h\Omega(v_k+l_1/2),$$

$$l_2=h\Omega(u_k+m_1/2),$$

$$m_3=-h\Omega(v_k+l_2/2),\quad l_3=h\Omega(u_k+m_2/2),$$

$$m_4=-h\Omega(v_k+l_3),\quad l_4=h\Omega(u_k+m_3).$$

We use v_k and l_j, $j=1,2,3$, in Eq. (8).

The mutually independent random variables $\xi_k^{(i)}$ and $\eta_k^{(i)}$ from Eqs. (7) and (8) are simulated in accordance with the laws

$$P(\xi=0)=2/3, \quad P(\xi=-\sqrt{3})=P(\xi=\sqrt{3})=1/6,$$

$$P(\eta=-1/\sqrt{12})=P(\eta=1/\sqrt{12})=1/2.$$

The method (7) has the second order of weak convergence with respect to the time step h. However, it usually gives more accurate results (especially under low noise level) and it is not essentially more complicated from the computational point of view than a standard weak method of order 2 (see details in [12]). To carry out our experiments, we do not take a more accurate method, e.g., a standard method of order 3 [9] or methods with errors $O(h^4+\varepsilon^6h^2)$, $O(h^4+\varepsilon^2h^3)$, $O(h^4+\varepsilon^4h^3)$, etc. of [12], because the method (7) ensures enough accuracy and speed of calculations for studying SR characteristics in our case. Therefore, we have no reasons to take a more accurate and, naturally, more complicated method (e.g., there are no fully Runge-Kutta schemes among the more accurate methods mentioned).

B. Evaluation of the signal-to-noise ratio

One of the main characteristics describing the SR phenomenon is the SNR. The SNR is a commonly used measure of the information content of the response of a system. The remarkable property of the SR phenomenon is the nonmonotonic behavior of SNR as a function of noise level. The function has a maximum and there is a noise level for which the system acts as a selective amplifier in some range of frequencies.

Here we use the following definition of the output SNR R for the constituent oscillator:

$$R=\frac{\alpha_1}{S_0(\Omega)},$$

where α_1 and $S_0(\omega)$ are from Eq. (6), i.e., R is the ratio of the signal power and the noise background at the frequency of the applied periodic force. The distinctions in SNR definition existing in the literature do not lead to qualitatively different results.

Let us introduce a sufficiently small interval of frequencies $[\Omega-\Delta\Omega,\Omega+\Delta\Omega]$ (''signal bin'') and approximate R by $\widetilde{R}$

$$\widetilde{R}=\frac{Q(\Omega,\Delta\Omega)-\Delta\Omega[S_0(\Omega-\Delta\Omega)+S_0(\Omega+\Delta\Omega)]}{[S_0(\Omega-\Delta\Omega)+S_0(\Omega+\Delta\Omega)]/2}, \tag{9}$$

where [see Eq. (5)]

$$Q(\Omega,\Delta\Omega)=F(\Omega+\Delta\Omega)-F(\Omega-\Delta\Omega)=\frac{2}{\pi}\int_0^{\infty}K(\tau)\cos\,\Omega\tau\frac{\sin\,\Delta\Omega\tau}{\tau}d\tau. \tag{10}$$

The term $Q(\Omega,\Delta\Omega)$ corresponds to the signal power and the noise background in the signal bin of width $2\Delta\Omega$ and the term $\Delta\Omega[S_0(\Omega-\Delta\Omega)+S_0(\Omega+\Delta\Omega)]$ is related to the noise background in the signal bin. As it is usually done, $S_0(\omega)$ is assumed to be a sufficiently slowly variating function. Our experiments proved this fact. This approximation of the SNR coincides in general with ones used in other papers on SR.

To calculate the SNR, the stationary conditions (2) for the solution to the system (1) must be fulfilled [as mentioned at the beginning of this section, the system (1) has a unique stationary solution]. Naturally, it is possible to ensure this requirement rigorously only for an infinite time. However, the solution may already have good for our aims (i.e., an error arising for this reason is not greater than the other errors in the experiment) stationary properties at a certain time moment T_s after the beginning of the SDE simulation. We find an appropriate value of T_s in our experiments by simulation of the expectation $EX(s)$ and the correlation function $EX_c(s)X_c(s+\tau)$ under various s.

Taking into account that $EX(s)$ is equal to zero in our case, let us write formulas omitting $EX(s)$. Due to the fact that the system is simulated on a finite time interval $[0,T_s+T]$, we approximate $Q(\Omega,\Delta\Omega)$ and $S_0(\omega)$ by $Q_T(\Omega,\Delta\Omega)$ and $S_{0_T}(\omega)$ correspondingly. Consequently, we calculate the value

$$\widetilde{R}_T=\frac{Q_T(\Omega,\Delta\Omega)-\Delta\Omega[S_{0_T}(\Omega-\Delta\Omega)+S_{0_T}(\Omega+\Delta\Omega)]}{[S_{0_T}(\Omega-\Delta\Omega)+S_{0_T}(\Omega+\Delta\Omega)]/2}, \tag{11}$$

which approximates $\widetilde{R}$ and therefore R. Note that we simulate the system (1) on the time interval $[0,T_s]$ to ensure the stationary properties (2) and on the time interval $[T_s,T_s+T]$ to evaluate $\widetilde{R}_T$.

The function $Q_T(\Omega,\Delta\Omega)$ from Eq. (11) is equal to [see Eq. (10)]

$$Q_T(\Omega,\Delta\Omega)=\frac{2}{\pi}\int_0^{T}K(\tau)\cos\,\Omega\tau\frac{\sin\,\Delta\Omega\tau}{\tau}d\tau=\frac{2}{\pi}E\left[X_c(T_s)\int_{T_s}^{T_s+T}X_c(t)\cos\,\Omega(t-T_s)\times\frac{\sin\,\Delta\Omega(t-T_s)}{t-T_s}dt\right].$$

We calculate it as

$$Q_T(\Omega,\Delta\Omega)=\frac{2}{\pi}EX_c(T_s)Z(T_s+T),$$

where $Z(t)$ obeys the subsidiary equation

$$dZ=X_c(t)\cos\,\Omega(t-T_s)\,\frac{\sin\,\Delta\Omega(t-T_s)}{t-T_s}dt, \quad Z(T_s)=0.$$

We simulate this equation together with the system (1) using the same method (see Sec. II A).

Let us consider $S_{0_T}(\omega)$ now. The spectral function $F(\omega)$ can be rewritten as

$$F(\omega)=\frac{1}{\pi}\int_0^\infty K(\tau)\frac{\sin\,\omega\tau}{\tau}d\tau$$

$$=\lim_{T\to\infty}\frac{1}{\pi}\int_0^T K(\tau)\left(\frac{1}{\tau}-\frac{1}{T}\right)\sin\,\omega\tau\,d\tau$$

because $(1/T)\int_0^T K(\tau)\sin\,\omega\tau\,d\tau$ goes to zero as T goes to infinity. Then, using the relation (cf. [14], Sec. 2.1)

$$\int_0^T K(\tau)\left(1-\frac{\tau}{T}\right)\cos\,\omega\tau\,d\tau$$

$$=\frac{1}{2T}E\left|\int_{T_s}^{T_s+T}X_c(t)e^{i\omega(t-T_s)}dt\right|^2,$$

we get

$$F(\omega)=\lim_{T\to\infty}\frac{1}{2\pi T}\int_0^\omega E\left|\int_{T_s}^{T_s+T}X_c(t)e^{i\widetilde{\omega}(t-T_s)}dt\right|^2 d\widetilde{\omega}.$$

Consequently,

$$\widetilde{S}(\omega)=\lim_{T\to\infty}\frac{1}{2\pi T}E\left|\int_{T_s}^{T_s+T}X_c(t)e^{i\omega(t-T_s)}dt\right|^2.$$

It is easy to prove (cf. [17]) that the function $\widetilde{S}(\omega)$ is exactly equal to $S_0(\omega)$ at the points $\omega\neq(2k-1)\Omega$, $k=1,2,\ldots$. Then we can take the function $S_T(\omega)$,

$$S_T(\omega)=\frac{1}{2\pi T}E\left|\int_{T_s}^{T_s+T}X_c(t)e^{i\omega(t-T_s)}dt\right|^2,$$

instead of $S_{0_T}(\omega)$ for evaluating $\widetilde{R}_T$ [see Eq. (11)] if $\Delta\Omega$ is sufficiently small and such that $S_T(\omega)$ has no sharp variations outside the intervals $[(2k-1)\Omega-\Delta\Omega,(2k-1)\Omega+\Delta\Omega]$, $k=1,2,\ldots$.

We calculate $S_T(\omega)$ as

$$S_T(\omega)=\frac{1}{2\pi T}E[Y_1^\omega(T_s+T)^2+Y_2^\omega(T_s+T)^2],\qquad(12)$$

where $Y_1^\omega(t)$ and $Y_2^\omega(t)$ obey the equations

$$dY_1^\omega=X_c(t)\,\cos\,\omega(t-T_s)\,dt,$$

$$dY_2^\omega=X_c(t)\,\sin\,\omega(t-T_s)\,dt,$$

$$Y_1^\omega(T_s)=Y_2^\omega(T_s)=0.\qquad(13)$$

We simulate Eq. (13) together with the system (1) by the same method.

In addition to the common errors (the error of numerical integration and the Monte Carlo error) arising in simulating SDEs by a weak method, we also have some error due to the finiteness of the time interval $[0,T_s+T]$ on which the system is simulated. Indeed, the estimators $Q_T(\Omega,\Delta\Omega)$ and $S_T(\omega)$ are biased ones for $Q(\Omega,\Delta\Omega)$ and $S_0(\omega)$ (see, e.g., [18]) since the errors $\rho_{Q_T}(\Omega,\Delta\Omega)=Q(\Omega,\Delta\Omega)-Q_T(\Omega,\Delta\Omega)$ and $\rho_{S_T}(\omega)=S_0(\omega)-S_T(\omega)$ are usually not equal to zero. However, taking the value of T to be large enough, we can ensure a sufficient smallness of these errors. To prove the smallness of these errors in comparison to other errors arising in our experiments, we check the variation of $\widetilde{R}_T$ with growing T. We consider T_s+T as a sufficiently long time if the variation of $\widetilde{R}_T$ with growing T is not greater than the other errors, e.g., the Monte Carlo error. The finiteness of the time interval also produces a loss of spectrum resolution [18], according to which we can calculate $S_T(\omega)$ only at $\omega=2\pi k/T$, $k=0,1,2,\ldots$. Because of time discretization, the highest frequency (the Nyquist frequency) that is possible to pick out is equal to $\omega_N=\pi/h$ [18], where h is a step of time discretization. The time step h in our experiments is such that frequencies $\omega\in[\Omega-\Delta\Omega,\Omega+\Delta\Omega]$ are much smaller than ω_N. To check that the taken time step h ensures the numerical integration error to be not greater than the other errors in our experiments, we carry out the repeated calculations of some points of $\widetilde{R}_T$ with the time step $h/2$.

Remark 1. As mentioned above, we simulate values such as $Ef(X(T_1))g(X(T_2))$ [e.g., $EX_c(s)X_c(s+\tau)$ or $Q_T(\Omega,\Delta\Omega)$] by the weak method. However, the existing theorems (cf. [9–12]) on weak convergence of a numerical method were proved for calculating an expectation such as $Ef(X(T))$. The proposition proved in [17] allows us to also simulate $Ef(X(T_1))g(X(T_2))$ by weak methods.

Remark 2. One can see that our procedure of calculating the SNR differs from the generally used one. The usual procedure contains the following steps (see, e.g., [19,4,5]): (a) simulation of SDE solutions $X(t)$ by a mean-square method (as a rule, the mean-square Euler method); (b) the random trajectories $X(t)$ obtained are considered as experimental statistical data, to which the fast Fourier transform is applied to calculate a random spectrum, and then, by averaging a number of segments and samples of the random spectrum, $S_T(\omega)$ is found; (c) the SNR is calculated on the basis of $S_T(\omega)$ [α_1 is found as a square under $S_T(\omega)$ in a signal bin].

According to our procedure, we use a weak method, which gives us an opportunity to calculate the needed values with a large integration step (e.g., 0.1 in our experiments) in comparison to the steps (0.005, 0.002, etc.) taken in previous works [19,4,5]; we simulate both the investigated system and the values needed for the approximation $\widetilde{R}_T$ of the SNR by the same numerical method. Because of its features, the proposed technique essentially saves CPU time, providing a sufficiently high accuracy.

C. Simulation of phase shifts

Another important characteristic of SR is a phase lag (phase shift) between the applied periodic force and the response. It was found for one-dimensional systems [20,21] that the phase shift also has a nonmonotonic (extremal) behavior with increasing noise.

As mentioned at the beginning of this section, the solution $X(t)$ to the system (1) is a periodic one with the period $2\pi/\Omega$ under any fixed initial phase $\varphi=\varphi_0$. By the definition of the periodic process [see Eq. (3)], we have

$$E[X(t)|\varphi=\varphi_0]=E[X(t+2\pi/\Omega)|\varphi=\varphi_0].$$

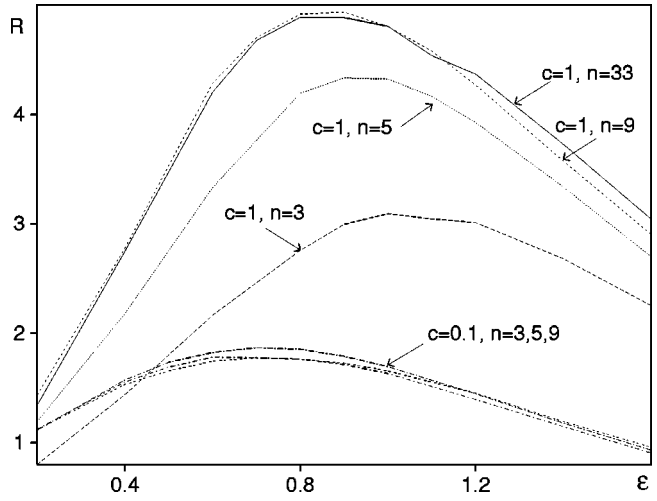

FIG. 1. Signal-to-noise ratio R versus noise level ε for the middle oscillators of the arrays (1) under the parameters $a=2.1078$, $b=1.4706$, $A=1.3039$, $\Omega=0.7301$, $T_s=6\pi/\Omega$, and $T=40\pi/\Omega$. The Monte Carlo error is less than 0.26.

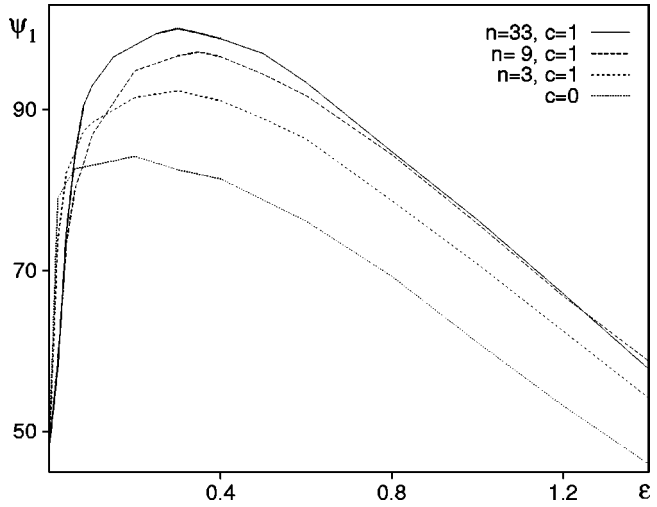

FIG. 3. Phase shifts ψ_1 (degrees) of the middle oscillators from the corresponding arrays described by Eq. (1). The parameters are the same as in Fig. 1 (with the exception of $T=8\pi/\Omega$). The Monte Carlo error is less than 1.7.

Then we can expand the conditional expectation $E[X(t)|\varphi=\varphi_0]$ in the Fourier series

$$E[X(t)|\varphi=\varphi_0]=\sum_{k=0}^{\infty}\beta_k\sin(k\Omega t+\varphi_0-\psi_k).$$

Analogously to [20,21], we are interested in the value of the phase shift ψ_1. It can be found for the constituent oscillator as

$$\psi_1=-\arctan[EZ_1(T_s+T)/EZ_2(T_s+T)], \tag{14}$$

where $Z_1(t)$ and $Z_2(t)$ obey the equations

$$dZ_1=X_c(t)\cos(\Omega t+\varphi_0)dt,$$

$$dZ_2=X_c(t)\sin(\Omega t+\varphi_0)dt, \quad Z_1(T_s)=Z_2(T_s)=0. \tag{15}$$

Here T_s is a time moment, after which we suppose, as above, that $X(t)$ has already good stationary properties; T is equal to $2\pi k/\Omega$, $k=1,2,\ldots$. Equations (15) are also simulated together with the system (1) by the same method (see Sec. II A).

III. RESULTS OF NUMERICAL EXPERIMENT

To carry out our experiments, we take the parameters of the system (1) just as in [4,5]: $a=2.1078$, $b=1.4706$, $A=1.3039$, and $\Omega=0.7301$. These values of parameters are such that in the absence of noise the oscillators are confined to a single well of the bistable potential, but small noise can induce significant hopping between wells [4].

The differential equations of Sec. II, needed for calculating the SNR and phase shifts, are simulated by the weak method from Sec. II A with the time step $h=0.1$, with the exception of the curves of Fig. 3 where we use $h=0.2$ for the noise level $\varepsilon=0-0.5$ (note that the main SR effect on the phase shifts is namely in this range of ε). To calculate the expectations, we simulate N_r independent realizations of $X(t)$. We take N_r equal to 4000 in all our experiments, except for the curve of Fig. 3 under $c=0$ and $\varepsilon=0.02-0.15$ for

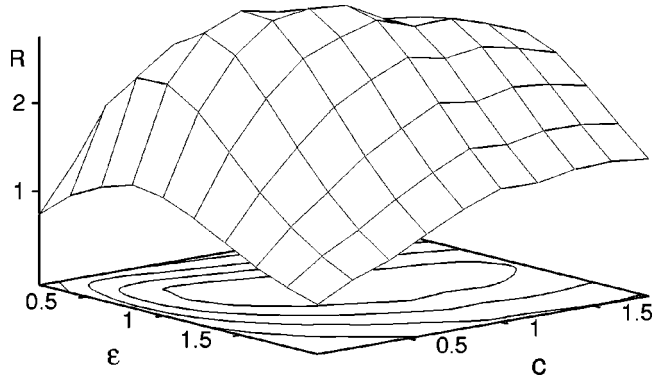

FIG. 2. Signal-to-noise ratio R versus noise level ε and coupling c for the middle oscillator of the array of three oscillators (1). The parameters are the same as in Fig. 1. The Monte Carlo error is less than 0.17.

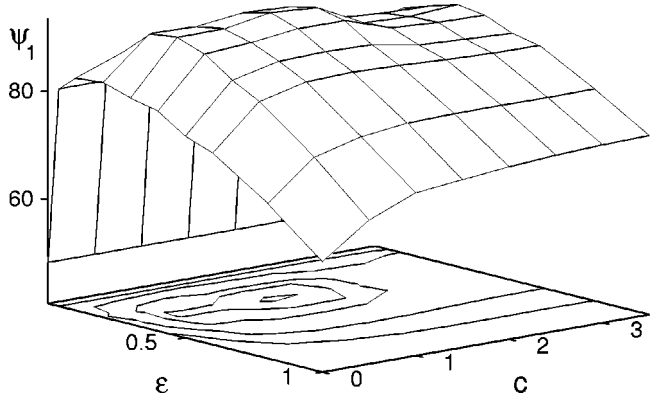

FIG. 4. Phase shifts ψ_1 (degrees) versus noise level ε and coupling c of the middle oscillator from the array of three oscillators. The parameters are the same as in Fig. 3. The Monte Carlo error is less than 1.9.

which $N_r = 10\,000$. All the errors arising in our experiments are smaller than or comparable to the Monte Carlo error (we control the errors by the repeated calculations of some points of the SNR and phase shifts under a smaller step h, greater N_r, and greater T). To simulate the needed random variables, we use a random generator of [22].

Figures 1 and 2 present the SNR behavior. The SNR is a nonmonotonic (extremal) function both of noise and of coupling. If the extremal behavior with increasing noise is a common feature of all systems connected with the SR phenomenon, the SNR extremal behavior with the growth of coupling is attributable to the AESR phenomenon. These results coincide with ones of [4,5]. We also come to the same conclusion as authors of previous papers that an increase of the array length n improves the SNR. According to Fig. 1, there is a length n_* of the array such that a further increase of the array length n does not lead to an improving SNR under each fixed coupling c.

Figures 3 and 4 show the extremal behavior of the phase shifts. Maxima of phase shifts appear at a lower noise level in comparison to the noise level corresponding to the SNR maxima. This fact corresponds to the results of [20] on a single oscillator. The phase shift is also a nonmonotonic function of coupling in the case of noisy coupled oscillators (see Fig. 4). An increase of the array length n improves the effect just as it affects the SNR (see Fig. 3).

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